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PROPER HOLOMORPHIC MAPPINGS FROM THE TWO-BALL TO THE THREE-BALL

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ABSTRACT. We prove that a proper mapping of the two ball in $\mathbf{C}^n$ into the three ball, which is C^2 on the closed two ball is equivalent to one of four normalized polynomial mappings. This improves the known result of Faran. The proof is basic using Taylor expansions.

INTRODUCTION

Let Ω and D be regions in $\mathbf{C}^n$ and $\mathbf{C}^k$ respectively. A holomorphic mapping $f\colon \Omega \to D$ is said to be a proper holomorphic mapping provided $f^{-1}(K)$ is a compact subset of Ω whenever K is a compact subset of D. We use the following notation. Let $\langle z, w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$ be the inner product of $z, w \in \mathbf{C}^n$, denote the corresponding norms by $\|z\|^2 = \langle z, z\rangle$ and let $B_n = \{z \in \mathbf{C}^n : \|z\| < 1\}$.

If $f\colon \Delta \to \Delta$ is a proper holomorphic mapping where $\Delta = B_1 =$ the unit disk in the complex plane, then it is easy to check that f is a finite Blaschke product. Alexander [1] showed that if $f\colon B_n \to B_n$ is a proper holomorphic mapping with $n \geq 2$ then f is a holomorphic automorphism of B_n onto B_n.

Let ϕ and ψ be holomorphic automorphisms of B_n and B_k respectively. Then $f\colon B_n \to B_k$ is a proper holomorphic mapping if and only if $g = \psi \circ f \circ \phi\colon B_n \to B_k$ is a proper holomorphic mapping. Under these conditions, we say that f is equivalent to g and we write $f \sim g$. Thus, since the holomorphic automorphisms of B_n are known, there is no loss in generality in assuming $f(0) = 0$. Further, if $\min\{\operatorname{rank} Df(z)\colon z \in B_n\}$, occurs at a point $z_0 = B_n$, we may assume without loss of generality (in our characterization of proper maps) that $z_0 = 0$. This is true because ϕ and ψ above can be chosen so that $\phi(0) = z_0$ and $\psi \circ f(z_0) = 0$. Then $g \sim f$, $g(0) = 0$ and

$$\operatorname{rank} Dg(0) = \operatorname{rank} D\psi(f(z_0)) \circ Df(z_0) \circ D\phi(0) = \operatorname{rank} Df(z_0)$$

because $D\psi(f(z_0))$ and $D\phi(0)$ are invertible.

We assume from this point on, that $f\colon B_n \to B_k$ is proper and holomorphic, $2 \leq n < k$ and that $f(0) = 0$.

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Webster [5], proved that if $F\colon \overline{B}_n \to \overline{B}_{n+1}$ ($\overline{A}$ is the closure of A), yields a C^3 immersion of ∂B_n into ∂B_{n+1}, $n \geq 3$, and F is holomorphic on B_n then $F(\overline{B}_n)$ is contained in an n-dimensional affine space. Thus, under our assumptions on f, if $3 \leq n, k = n+1$ and f extends to a C^3 mapping of $\overline{B}_n$ into $\overline{B}_{n+1}$ then Webster's result says that f is equivalent to the mapping $(z_1, z_2, \ldots, z_n) \to (z_1, z_2, \ldots, z_n, 0)$. Faran [3], then showed that if $n = 2$, $k = 3$ and f extends to a C^3 mapping of $\overline{B}_2$ into $\overline{B}_3$ then f is equivalent to one of the following four mappings.

$$(1)\qquad \begin{aligned} &\text{(i)} && (z, w) \to (z, w, 0), \\ &\text{(ii)} && (z, w) \to (z, zw, w^2), \\ &\text{(iii)} && (z, w) \to (z^2, \sqrt{2}zw, w^2), \text{ or} \\ &\text{(iv)} && (z, w) \to (z^3, \sqrt{3}zw, w^3). \end{aligned}$$

In [2], the present authors showed that Webster's and Faran's results were true under the weaker assumption that f extends to a C^2 mapping of $\overline{B}_n \to \overline{B}_{n+1}$ ($n \geq 3$ for Webster's result and $n = 2$ for Faran's result). We further showed that if f extends to a C^2 mapping on $\overline{N \cap B_n}$ where N is an open subset of $\mathbf{C}^n$ with $N \cap \partial B_n \neq \phi$, $k \leq n(n+1)/2$ and for each $u \in N \cap \partial B_n$, the set of vectors

$$(2)\qquad \{f(u)\} \cup \{Df(u)(v_j)\colon 2 \leq j \leq n\} \cup \{D^2 f(u)(v_l, v_p)\colon (l, p) \in K\}$$

is linearly independent then f is a rational function. Here $u = (u_1, u_2, \ldots, u_n)$, $u_1 \neq 0$ and $v_2 = (\bar{u}_2, -\bar{u}_1, 0, \ldots, 0)$, $v_3 = (\bar{u}_3, 0, -\bar{u}_1, 0, \ldots, 0), \ldots,$ $v_n = (\bar{u}_n, 0, \ldots, 0, -\bar{u}_1)$, $2 \leq j \leq n$, and K is an appropriate set of $k - n$ pairs of integers (l, p). Thus (2) is a collection of k linearly independent vectors in $\mathbf{C}^k$. We further showed that $f = P/S$ where P is a (vector-valued) polynomial of degree $\leq 2k - n - 1$ and S is a polynomial of degree $\leq \deg P - 1$ (assuming $f(0) = 0 = P(0)$). We conjectured that Webster's result could be improved to state that if $2 < n < k \leq 2n - 2$ and f extends to a C^2 mapping of $\overline{B}_n$ into $\overline{B}_k$ then f is equivalent to the trivial mapping

$$(z_1, z_2, \ldots, z_n) \to (z_1, z_2, \ldots, z_n, 0, 0, \ldots, 0).$$

This would be the best possible result (as far as the size of k is concerned) since the mapping $f\colon B_n \to B_{2n-1}$ given by

$$f(z_1, z_2, \ldots, z_n) = (z_1, z_2, \ldots, z_{n-1}, z_1 z_n, z_2 z_n, \ldots, z_n^2)$$

is a proper holomorphic mapping. Faran [4] has shown that this conjecture is true under the stronger assumption that f extends to a holomorphic mapping from an open set $N \subset \mathbf{C}^n$, $\overline{B}_n \subset N$, into $\mathbf{C}^k$.

Faran's nice results concerning proper holomorphic maps from B_2 into B_3 [3] were obtained by using moving frames and doing some long and tedious computations in projective space. In this paper, we obtain Faran's results by elementary means starting from our results in [2] which were also obtained by elementary means. We believe the method used here (i.e. minimizing $\|Dg(0)\|$ over all g equivalent to f with $g(0) = 0$) may be useful for characterizing the proper maps $f: B_n \to B_k$ for other n and k. As described above, our main result is as follows.

Main Theorem. *If $f: B_2 \to B_3$ is a proper holomorphic mapping that extends to a C^2 mapping of $\overline{B}_2$ into $\overline{B}_3$ then f is equivalent to one of the four mappings given in* (1).

PRELIMINARY RESULTS

In [2, Theorem 3], we showed that if $n = 2$, $k = 3$ and f extends to be a C^2 map of $\overline{B}_2$ into $\overline{B}_3$ and f does not satisfy the linear independence conditions of our main theorem then f is equivalent to (1)(i). Therefore, we need only consider

$$f = \frac{1}{S}P, \quad P(0) = 0, \quad S(0) = 1, \quad \deg S \le \deg P - 1 \le 2. \tag{3}$$

The fact that $\deg S \le \deg P - 1$ follows from the fact that

$$\left\langle P(\lambda u), P\left(\frac{1}{\bar{\lambda}}u\right)\right\rangle = S(\lambda u)\overline{S\left(\frac{1}{\bar{\lambda}}u\right)} \tag{4}$$

when $\|u\| = 1$, λ a complex number, upon equating coefficients of the highest power of λ.

We will always assume, unless otherwise stated, that λ is complex, $u \in \mathbf{C}^2$, $\|u\| = 1$ and $v = (\bar{u}_2, -\bar{u}_1)$ so that $\|v\| = 1$ and $\langle u, v\rangle = 0$. For any holomorphic function $h(z, w)$, $(z, w) \in B_2$, we define $h^*(z, w) = \overline{h(\bar{w}, -\bar{z})}$ so h^* is also holomorphic and we note that

$$h^{**}(z, w) = h(-z, -w). \tag{5}$$

Since f is rational, it can be expanded about almost every boundary point. From [2, Lemma 1], it follows that $\langle f(u), f(u + \lambda v)\rangle \equiv 1$. Thus,

$$\langle P(u), P(u + \lambda v)\rangle = S(u)\overline{S(u + \lambda v)}. \tag{6}$$

Hence we see that if H is a vector-valued homogeneous polynomial consisting of the highest degree terms of P, then

$$\langle P(u), H(v)\rangle = 0 \tag{7}$$

because this is the coefficient of the highest degree terms in $\bar{\lambda}$. Since $\langle P(z, w), \overline{H^*(z, w)}\rangle$ is holomorphic for $(z, w) \in B_2$ (in fact for all (z, w) since P and H^* are polynomials) and equals zero when $\|(z, w)\| = 1$, we conclude

$$0 = \langle P(z, w), \overline{H^*(z, w)}\rangle = \sum_{j=1}^{3} P_j(z, w)H_j^*(z, w) \tag{8}$$

for all (z, w).

Now let L consist of the linear terms of P. By proper choice of coordinates, we may assume $L(z, w) = (az + cw, bw, 0)$. In fact, since there exist perpendicular, nonzero vectors $(z, w), (\bar{w}, -\bar{z})$ such that $\langle L(z, w), L(\bar{w}, -\bar{z})\rangle = 0$, we may find unitary maps U and V on $\mathbf{C}^2$ and $\mathbf{C}^3$ respectively such that $V \circ f \circ U$ has linear part $(az, bw, 0)$ with $1 \geq a \geq 0$ and $1 \geq b \geq 0$. We will assume $f = P/S$ has this form when it is convenient to do so.

The difficulty in proving f is equivalent to one of the mappings (1) is that it is not always easy to recognize f as being $\psi \circ g \circ \phi$ where g is one of the mappings (1) and ϕ and ψ are biholomorphic automorphisms of B_2 and B_3 respectively. Therefore, we first consider briefly the effect of forming $\psi \circ f \circ \phi$. Let $Z \in B_2$ and $W \in B_3$, $r > 1$, u fixed, $u \in \partial B_2$ and $R = \|f(ru)\|$. We define ϕ and ψ so that $\phi(0) = ru$ and $\psi(f(ru)) = 0$.

$$\phi(Z) = \frac{1}{1 + r\langle Z, u\rangle}\left(\langle Z + ru, u\rangle u + \sqrt{1 - r^2}(Z - \langle Z, u\rangle u)\right),$$

$$\psi(W) = \frac{1}{1 - \langle W, f(ru)\rangle}\left(\left\langle W - f(ru), \frac{1}{R}f(ru)\right\rangle \frac{f(ru)}{R} + \sqrt{1 - R^2}\left(W - \left\langle W, \frac{1}{R}f(ru)\right\rangle \frac{f(ru)}{R}\right)\right).$$

Now write $Z = \alpha u + \beta v$. Then

$$\begin{aligned}\phi(\alpha u + \beta v) &= \frac{\alpha + r}{1 + \alpha r}u + \frac{\sqrt{1 - r^2}}{1 + \alpha r}\beta v \\ &= ru + (1 - r^2)\alpha u + \sqrt{1 - r^2}\beta v + o_1(\alpha, \beta), \\ f \circ \phi(\alpha u + \beta v) &= f(ru) + (1 - r^2)\alpha Df(ru)(u) \\ &\quad + \sqrt{1 - r^2}\beta\, Df(ru)(v) + o_2(\alpha, \beta)\end{aligned}$$

and

$$\begin{aligned}\psi \circ f \circ \phi(\alpha u + \beta v) &= \frac{1 - r^2}{1 - R^2}\alpha\langle Df(ru)(u), U\rangle U \\ &\quad + \frac{1 - r^2}{\sqrt{1 - R^2}}\alpha(Df(ru)(u) - \langle Df(ru)(u), U\rangle U \\ &\quad + \frac{\sqrt{1 - r^2}}{1 - R^2}\beta\langle Df(ru)(v), U\rangle U \\ &\quad + \frac{\sqrt{1 - r^2}}{\sqrt{1 - R^2}}\beta(Df(ru)(v)) - \langle Df(ru)(v), U\rangle U) \\ &\quad + o_3(\alpha, \beta)\end{aligned} \tag{10}$$

where $U = f(ru)/R$ and we have used the fact that

$$\frac{1}{1 - \langle f(ru) + O(\alpha, \beta), f(ru)\rangle} = \frac{1}{1 - R^2 + O(\alpha, \beta)} = \frac{1}{1 - R^2} + O(\alpha, \beta).$$

Here, $\|o_j(\alpha,\beta)\|/\|(\alpha,\beta)\| \to 0$ and $O(\alpha,\beta)/\|(\alpha,\beta)\|$ is bounded as $(\alpha,\beta) \to (0,0)$. Set $g_{ru} = \psi \circ f \circ \phi$. Assume a sequence of $r_n \to 1$ and u_n, $\|u_n\| = 1$ are chosen so that $r_n u_n \to u$ and $\lim_{n\to\infty} g_{r_n u_n} \to g$ uniformly on compact subsets of B_2. Then clearly g is holomorphic, $g(0) = 0$ and $\|g(Z)\| < 1$ for all $Z \in B_2$. Further,

$$\frac{1-r_n^2}{1-R_n^2} \to \frac{1}{\langle Df(u)(u), f(u)\rangle} \quad (\text{when } R_n = \|f(r_n u_n)\|)$$

so the first term on the right in (10) has limit $\alpha f(u)$. Since the first and third terms on the right lie in the same one dimensional space perpendicular to the second and fourth terms, the third term must have limit zero (if the limit is l, then

$$\|g(\alpha u + \beta v)\| \geq \|\alpha f(u) + \beta l\| = |\alpha + c\beta| > \sqrt{|\alpha|^2 + |\beta|^2}$$

for some choice of α and β unless $c = 0$). It follows that

$$\frac{1}{\sqrt{1-R_n^2}}\langle Df(r_n u_n)(v_n), f(r_n u_n)\rangle \to 0 \quad \text{as } n \to \infty.$$

Since $\|Df(u)(v)\|^2 = \langle Df(u)(u), f(u)\rangle$, [2, Lemma 1(b)], the fourth term has limit $\beta Df(ru)(v)/\|Df(ru)(v)\|$ while the second term clearly has limit zero. Thus, it is easy to see that g is a linear isometry. In fact, the same argument shows that even without the assumption $\lim_{n\to\infty} g_{r_n u_n} = g$ uniformly on compact subsets of B_2, we still have $Dg_{r_n u_n}(0) \to L$ where L is a linear isometry. Since holomorphic automorphisms are always of the form (9) up to composition with an appropriate unitary map, we have proved the following lemma.

Lemma 1. *Let f be a given proper holomorphic mapping of B_n into B_k that extends to a C^1 mapping of $\overline{B}_n$ into $\overline{B}_K$ with $f(0) = 0$. Let $g_{\psi,\phi} = \psi \circ f \circ \phi$ where ϕ and ψ are holomorphic automorphisms of B_n and B_k respectively with $g_{\psi,\phi}(0) = 0$. If ϕ varies so that $\phi(0) \to \partial B_n$ then $Dg_{\psi,\phi}(0) \to L$ where L is a linear isometry. Further, there exists ϕ_0, ψ_0 so that $\|Dg_{\psi_0,\phi_0}(0)\| \leq \|Dg_{\psi,\phi}(0)\|$ for all ϕ and ψ as above.*

The result is clearly true for general n and k because the dimension k of the range space does not enter into the proof while the proof for the domain B_2 can be applied, viewing the vector v in the proof as being any unit vector perpendicular to u.

It is clear that we may choose f so that

$$\|Df(0)\| \leq \|Dg_{\psi,\phi}(0)\| \tag{11}$$

for all ϕ, ψ as above. Our proof then consists of the following cases.

Case 1. If $Df(0) = 0$ then f is equivalent to (1)(iii) or (1)(iv).

Case 2. If $Df(0)$ can be chosen to have rank 1, then f is equivalent to (1) (ii).

Case 3. If $\|Df(0)(\alpha,\beta)\| = \rho\|(\alpha,\beta)\|$ for all $(\alpha,\beta) \in \mathbf{C}^2$ and some constant $\rho > 0$ then $\rho = 1$ and f is equivalent to (1)(i).

Case 4. Under the condition (11), if $Df(z, w)$ always has maximal rank then $\|Df(0)(\alpha, \beta)\|/\|(\alpha, \beta)\|$ is constant, $(\alpha, \beta) \neq (0, 0)$.

The following lemma will be useful. Suppose $u \in \mathbf{C}^n$, $\|u\| = 1$, and let $H = \{Z \in \mathbf{C}^n : \langle Z, u\rangle = 0\}$. Let $A \in B_k$ satisfy $\|A\| = 1$.

Lemma 2. *Suppose $f: B_n \to B_k$ is proper holomorphic and $f(\lambda u) = h(\lambda)A$, $|\lambda| < 1$. Then $h(\lambda)$ is a finite Blaschke product and*

$$f(\lambda u + Z) = h(\lambda)A + p(\lambda, Z), \qquad |\lambda|^2 + \|Z\|^2 < 1, \ Z \in H,$$

where $\langle p(\lambda, Z), A\rangle = 0$.

Proof. The fact that h is a finite Blaschke product follows from the fact that $\|f(\lambda u)\| \to 1$ as $|\lambda| \to 1$. Now set $p(\lambda, Z) = f(\lambda u + Z) - h(\lambda)A$ where $Z \in H$ and $|\lambda|^2 + \|Z\|^2 < 1$. Now apply [2, Lemma 1] to conclude $\langle D^l f(\lambda u)(Z), A\rangle = 0$ for $l = 1, 2, \ldots$. This completes the proof.

Now assume $n = 2$, $k = 3$, $P = L + Q + T$, $S = 1 + l + q$ where L, Q and T are homogeneous polynomials of degree 1, 2 and 3 respectively and l and q are homogeneous of degree 1 and 2 respectively. Also, as remarked earlier, we may assume $L(z, w) = (az, bw, 0)$ where $1 \geq a \geq b \geq 0$. With this notation, we have the following lemma.

Lemma 3. *For all $(z, w) \in \mathbf{C}^2$,*

$$\langle P(z, w), T(\bar{w}, -\bar{z})\rangle \equiv 0. \tag{12}$$

Further, if $a \geq b > 0$, we may assume

$$T(z, w) = \left(\frac{z}{a}q(z, w), \frac{w}{b}q(z, w), T_3(z, w)\right) \tag{13}$$

where T_3 has one of the following forms

$$\begin{aligned} &\text{(i)} \quad T_3(z, w) = rzq(z, w) \text{ or} \\ &\text{(ii)} \quad T_3(z, w) = rwl_1 l_2^* \text{ or} \\ &\text{(iii)} \quad T_3(z, w) = rzl_1^* l_2^* = rzq^* \end{aligned} \tag{14}$$

where $r = \sqrt{1/b^2 - 1/a^2}$ and $q = l_1 l_2$ with l_1 and l_2 linear.

Proof. If $u = (z, w)$, $\|u\| = 1$, then (12) is the same as (7). However, (12) is a holomorphic function for all (z, w) and hence (12) is true for all $(z, w) \in \mathbf{C}^2$. Equating coefficients of λ^2 in (4) yields $a\bar{z}T_1(z, w) + b\bar{w}T_2(z, w) = q(z, w)$ when $\|(z, w)\| = 1$. For $z \neq 0$, replace $\bar{z}$ by $(1 - w\bar{w})/z$ to obtain

$$\frac{a}{z}T_1(z, w) + \bar{w}\left(bT_2(z, w) - a\frac{w}{z}T_1(z, w)\right) = q(z, w), \qquad \|(z, w)\| = 1. \tag{15}$$

Since both sides of (15) are holomorphic in z when $|z| > 0$ it follows that equality holds for all $(z, w) \in \mathbf{C}^2$, $z \neq 0$. This implies $T_1 = zq/a$ and

$T_2 = awT_1/bz = wq/b$. The highest degree terms in (7) are $\langle T, \overline{T^*}\rangle \equiv 0$. Thus,

$$\frac{1}{a^2}zwqq^* - \frac{1}{b^2}zwqq^* + T_3(z,w)T_3^*(z,w) = 0$$

so

$$T_3(z,w)\overline{T_3(\overline{w}, -\overline{z})} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right) zwq(z,w)q^*(z,w) = r^2 zwl_1l_2l_1^*l_2^*.$$

Hence, either l_1 and l_2 are factors of T_3 or l_1 and l_2^* are factors of T_3 or l_1^* and l_2^* are factors of T_3. Further, either z or w is a factor of T_3. Checking all the possibilities, we see that (14)(i), (ii) and (iii) are the only choices of T_3 (except for interchanging l_1 and l_2) that yield the correct value for $T_3T_3^*$.

Proof of the Main Theorem. Now assume $Df(0) \equiv 0$ so that $f = (Q+T)/S$. Since the left side of (4) has no λ^2 terms, it follows that $q \equiv 0$ so $s = 1 + l$. Now assume $T \not\equiv 0$. By replacing f by $f \circ U$ where U is unitary, if necessary, we may assume

$$f(z,w) = \frac{1}{1+l}\sum_{k=2}^{3}\sum_{j=0}^{k} z^{k-j}w^j A_{k-j,j},$$

$A_{30} \neq 0$ and $A_{03} \neq 0$. The equality $\langle P(u), P(u)\rangle = |1+l|^2$, $\|u\| = 1$ implies that $\langle A_{30}, A_{03}\rangle = 0$ (i.e. the $z_3\overline{w}^3$ term must drop out). Set $E_1 = A_{30}/\|A_{30}\|$ and $E_3 = A_{03}/\|A_{03}\|$ and let E_2 be a unit vector in $\mathbf{C}^3$ that is orthogonal to both E_1 and E_3 so that $\{E_1, E_2, E_3\}$ is an orthonormal basis for $\mathbf{C}^3$. We now use

$$(16) \qquad \langle P(z,w), P(z,w)\rangle = |1 + l(z,w)|^2, \qquad \|(z,w)\| = 1,$$

and

$$(17) \qquad \langle Q(z,w), \overline{T^*(z,w)}\rangle = 0.$$

From (16), we see that $\langle A_{30}, A_{02}\rangle = 0$ and $\langle A_{03}, A_{20}\rangle = 0$. From (17), $\langle A_{30}, A_{20}\rangle = 0$ and $\langle A_{03}, A_{02}\rangle = 0$. Now using (16) again, we get $\langle A_{30}, A_{11}\rangle = 0$ and $\langle A_{03}, A_{11}\rangle = 0$. Thus, $Q(z,w) = h(z,w)E_2$ where h is a scalar homogeneous polynomial of degree 2. By use of a unitary map in $\mathbf{C}^2$ we may assume $A_{02} = 0$. This may necessitate redefining A_{03} and A_{30} but E_2 will remain as before with $\langle A_{30}, E_2\rangle = 0 = \langle E_2, A_{03}\rangle$. Now setting $z = 0$, we see that (aside from a constant scalar of modulus 1), $f(0,w) = w^3E_3$. Using Lemma 2, we conclude $l(z,w) \equiv 0$. By proper choice of coordinates we now may assume $f(z,w) = (T_1, T_2 + Q_2, w^3)$ where z is a factor of T_1, T_2 and Q_2 and w is a factor of T_2. Using (17), we see that either $T_2 \equiv 0$ or $Q_2 \equiv 0$. Now $Q_2 \equiv 0$ implies $T_1 = z^3$ (to see this, set $w = 0$) but then $|z|^6 + |T_2|^2 + |w|^6$ cannot be 1 on ∂B_2 so we must have $T_2 \equiv 0$. Now $Q_2 = z(\alpha z + \beta w)$ for some α and β. Using (16), we see that $T_1 = cz^kw^{3-k}$ for some c and k, $1 \le k \le 3$, and either $\alpha = 0$ or $\beta = 0$. It is routine to check that $k = 3$, $|c| = 1$, $\alpha = 0$, $|\beta| = \sqrt{3}$ is the only possiblity. That is f is equivalent to (1)(iv).

Now assume $Df(0) = 0$ and $T \equiv 0$. From (4), it is easy to see that $S \equiv 1$ and $f = Q$. In the same notation as above, it is trivial to see that $\langle A_{20}, A_{02}\rangle = 0 = \langle A_{20}, A_{11}\rangle = \langle A_{11}, A_{02}\rangle$. Further, it then readily follows that f is equivalent to (1)(iii).

We have completed Case 1 of the previous section.

We now proceed to Case 2 assuming $Df(0)$ has rank 1. By proper choice of coordinates, $L(z) = (az, 0, 0)$, $1 \geq a > 0$. Since $T_1 a\bar{z} = q$ (i.e. from (4), the coefficients of λ^2 must be equal), $|z|^2 + |w|^2 = 1$ and both sides are holomorphic in w, the equality persists for all (z, w). Hence $T_1 \equiv 0 \equiv q$. Thus,

$$f = \frac{1}{1+l}(az + Q_1, Q_2 + T_2, Q_3 + T_3).$$

Write

$$P(z) = azA_{10} + \sum_{k=2}^{3}\sum_{j=0}^{k} z^{k-j}w^j A_{k-j,j}.$$

If $T \equiv 0$, then using (4) it is easy to see that $Q_1 \equiv 0 \equiv l$. Setting $z = 0$, it easily follows that $\|A_{02}\| = 1$ so by proper choice of coordinates using Lemma 2, we may assume $Q_3 = w^2$. It is then straightforward to conclude that $Q_2 = cz^k w^{2-k}$, $k = 1$ or $k = 2$. Then it follows readily that $k = 1$, $a = 1$ and $|c| = 1$ so f is equivalent to (1)(ii).

Now assume $T \not\equiv 0$. As before, $\langle A_{30}, A_{03}\rangle = 0$ and after possibly composing with a unitary map, we may assume z is a factor of T_2 and w is a factor of T_3. Set $T_2 = zl_1l_2$ where l_1 and l_2 are linear. Using (7), we conclude $T_2T_2^* + T_3T_3^* = 0$. It then follows that either $T_3 = wl_1l_2$ or $T_3 = wl_1^*l_2^*$. Now fix $u = (z, w)$, $\|u\| = 1$. Using [2, Theorem 2], there exist constants c_1, c_2, c_3 (not all 0) such that $c_1P_1(\lambda u) + c_2P_2(\lambda u) + c_3P_3(\lambda u) = 0$, $|\lambda| < 1$. Here we may assume c_1, c_2 and c_3 vary continuously with $u = (z, w)$. Divide by λ and let $\lambda \to 0$ to see that $c_1 = 0$. Then $c_2T_2(u) + c_3T_3(u) = 0$ and $c_2Q_2(u) + c_3Q_3(u) = 0$.

Assuming $T_3 = wl_1l_2$, we may take $c_3 = -z$, $c_2 = w$. Then it follows that w is a factor of Q_3 and z is a factor of Q_2. In fact, $P_2/z = P_3/w$. Then the map

$$h(z, w) = \frac{1}{1+l}\left(az + Q_1, \frac{P_2}{z}\right)$$

is a proper mapping with $h(0, 0) = (0, 0)$. This implies that h is unitary so $l \equiv 0$, $Q_1 \equiv 0$, $a = 1$. $P_2/z = w$ (except for a possible rotation). In this case, f is equivalent to (1)(ii).

Now assume $T_3 = wl_1^*l_2^*$. We may take $c_2 = -Q_3(u)$, $c_3 = Q_2(u)$ so that $Q_3zl_1l_2 = Q_2wl_1^*l_2^*$. It is readily seen that l_1^* is not a multiple of l_1. If l_1^* is a multiple of l_2 then l_2^* is a multiple of l_1 and we actually have $T_3 = wl_1l_2$ again. If neither l_1^* nor l_2^* is a multiple of z, then $wl_1^*l_2^*$ is a factor of Q_3 and this is impossible since $\deg Q_3 = 2$. Thus, we may assume $l_2 = w$ and $l_2^* = -z$. Then l_1 is a factor of Q_2 and l_1^* is a factor of Q_3. It then follows that Q_2 and Q_3 have a common factor l_3 and $P_2 = l_1(l_3 + zw)$ and $P_3 = l_1^*(l_3 + zw)$. If $l_1 = \alpha z + \beta w$, then $|l_1|^2 + |l_1^*|^2 = |\alpha|^2 + |\beta|^2$ when $|z|^2 + |w|^2 = 1$ and it follows that

$$h(z, w) = \frac{1}{1+l}\left(P_1, \frac{P_2}{\sqrt{|\alpha|^2 + |\beta|^2 l_1}}\right)$$

is proper with $h(0, 0) = 0$. This again means h is unitary so (up to a rotation $w \to we^{i\phi}$), $h(z, w) = (z, w)$. Then,

$$a = 1, \quad Q_1 = zl, \quad \frac{Q_2 + T_2}{1+l} = wz \quad \text{and} \quad \frac{Q_3 + T_3}{1+l} = w^2.$$

Hence f is equivalent to (1)(ii). This completes Case 2.

Now assume $\|Df(0)(\alpha, \beta)\| = \rho\|(\alpha, \beta)\|$ for all $(\alpha, \beta) \in \mathbf{C}^2$ where $0 < \rho \le 1$. In this case, we may take $L(z, w) = (az, aw, 0)$, $a = \rho$. By Lemma 3, $T_3 = 0$ and $T_1 = zq/a$, $T_2 = wq/a$. Using (7), we see that $\langle Q, \overline{T}^* \rangle \equiv 0$ so $Q_1 wq^*/a - Q_2 zq^*/a \equiv 0$. Therefore $Q_2 = wQ_1/z$.

We now see that $f_2 = wf_1/z$ and it follows that $(z, w) \to (f_1/z, f_3)$ is a proper map of B_2 into B_2 hence is of degree 1 (or is constant). However the degree of this map is at least as large as $\deg Q_3/S = 2$ unless $Q_3 \equiv 0$. Therefore, $Q_3 \equiv 0$, $a = 1$ and f is equivalent to (1)(i).

One case remains. We now assume $Df(0)$ has rank 2,

$$\|Df(0)(\alpha, \beta)\| / \|(\alpha, \beta)\|$$

is not constant, $(\alpha, \beta) \neq (0, 0)$, and that $\|Df(0)\| \le \|Dg_{\psi\phi}(0)\|$ for all ψ, ϕ holomorphic automorphisms of B_3 and B_2 respectively. Let ϕ and ψ be given by (9). Then,

$$D\phi(0) = I + O(r^2) \quad \text{and} \quad D\psi(f(ru)) = I + O(r^2)$$

where I is the identity on the appropriate space. Therefore,

$$D\psi \circ f \circ \phi(0) = Df(ru) + o(r).$$

We may assume $L(z, w) = (az, bw, 0)$ where $0 < b < a \le 1$. We wish to show that the minimal norm property implies that the quadratic part Q of f, has the form $Q_1(u) = au_1 l(u) + a_{02}u_2^2$ for some a_{02}. To that end, we first show that we need only consider (z, w), $|z|^2 + |w|^2 = 1$ with $|w| < \varepsilon$ where ε is small and positive. Set $L_{ru} = Df(ru)$. We know $a = \|L\| \le \|L_{ru}\|$. We have $L_{ru}(z, w) = (az + O(r), bw + O(r), O(r))$. Let $\varepsilon > 0$ and $|w| \ge \varepsilon$. Clearly, there exists $r(\varepsilon) > 0$ and $\delta > 0$ such that $r < r(\varepsilon)$ implies

$$\begin{aligned}\|L_{ru}(z, w)\|^2 &= a^2|z|^2 + b^2|w|^2 + O(r) < a^2 - \delta \\ &< \|L\|^2 \text{ when } |z|^2 + |w|^2 = 1, \quad |w| \ge \varepsilon.\end{aligned}$$

Hence we assume $|w| < \varepsilon$, ε to be chosen later. Then

$$\begin{aligned}\|L_{ru}(z,w)\|^2 &= \left|z\frac{\partial f_1}{\partial z}(ru)\right|^2 + O_1(\varepsilon) + O_2(r^2)\\ &= \left|\left[(1+rl(u))\left(az + r\frac{\partial Q_1}{\partial z}(u)z\right) - aru_1 l(z,0)\right](1-2rl(u))\right|^2\\ &\quad + O_1(\varepsilon) + O_2(r^2)\\ &= |z|^2(a^2 + 2ar\mathrm{Re}\left[\frac{\partial Q_1}{\partial z}(u) - 2al(u) + al(0,u_2)\right]\\ &\quad + O_1(\varepsilon) + O_2(r^2) < a^2\end{aligned}$$

for sufficiently small ε and r for appropriate choice of u unless $\partial Q_1(u)/\partial z = 2al(u) - al(0,u_2)$. That is,

$$Q_1(u) = au_1 l(u) + a_{02}u_2^2 \tag{18}$$

for some a_{02}.

Now assume $q = l_1 l_2$ and consider the fifth degree terms of (7). By Lemma 3, there are three possibilities for T_3. Assume first $T_3 = rzl_1l_2$ and $r = \sqrt{1/b^2 - 1/a^2}$. We have

$$Q_1\frac{w}{a}l_1^* l_2^* - Q_2\frac{z}{b}l_1^* l_2^* + Q_3 r l_1^* l_2^* = 0.$$

Therefore, w is a factor of Q_2 so $Q_2 = wl_3$. Then $Q_1 = azl_3/b - arQ_3$.

If we replace f by $U\circ f$ where U is the unitary map

$$\begin{pmatrix} \frac{b}{a} & 0 & br \\ 0 & 1 & 0 \\ -br & 0 & \frac{b}{a}\end{pmatrix}$$

then f has the form

$$\frac{1}{S}\left(bz + zl_3 + \frac{z}{b}q\,,\, bw + wl_3 + \frac{w}{b}q\,,\, -abrz - arzl_3 + \left(\frac{b}{a} + abr^2\right)Q_3\right).$$

Since $f_1/z = f_2/w$, the map $(z,w)\to(f_1/z, f_3)$ is proper and therefore of degree 1. This means S, P_1, P_2 and P_3 have a common factor so we may take $S = 1 + l$ and $f = (bz + zl_4, bw + wl_4, -abrz)/(1+l)$. Using (4) and equating the coefficients of λ we get $l_4 = l/b$.

If $b = 1$, then $a = 1$ and $f = (z, w, 0)$. If $b < 1$, setting $z = 0$ we see that $|(b + \frac{1}{b}l(0,w))/(1 + l(0,w))| = 1$ when $|w| = 1$. Hence by Lemma 2, $l(z,w) = l(0,w)$ and in fact after a possible rotation, $l(z,w) = bw$. Thus,

$$f = \left(z\frac{b+w}{1+bw}, w\frac{b+w}{1+bw}, \frac{-\sqrt{a^2-b^2}z}{1+bw}\right).$$

Setting $w = 0$, we see that $a = 1$ and replacing f by $V\circ f$ where

$$V = \begin{pmatrix} b & 0 & -\sqrt{1-b^2} \\ 0 & 1 & 0 \\ \sqrt{1-b^2} & 0 & b\end{pmatrix}$$

we get

$$f = \left(z, \frac{w(b+w)}{1+bw}, \frac{\sqrt{1-b^2}zw}{1+bw}\right).$$

It is easy to see that $Df(0,w)$ has rank 1 for an appropriate choice of w and this case has already been considered.

Now consider $T_3 = rwl_1 l_2^*$.

Using (7) as before, we get

$$Q_1 \frac{w}{a} l_1^* l_2^* - Q_2 \frac{z}{b} l_1^* l_2^* + Q_3 rzl_1^* l_2 = 0.$$

Therefore, we may assume $Q_1 = azl_3$ (if z is a factor of l_2^* then w is a factor of l_2 and this case would be identical to the previous Case 1). Thus, $wl_3 l_2^* - Q_2 l_2^*/b + rQ_3 l_2 = 0$ so $Q_3 = l_2^* l_4$. It follows that $Q_2 = brl_2 l_4 + bwl_3$. Due to our assumption on the minimality of $\|L\| = \|Df(0)\|$, we have $azl_3 = azl + a_{02}w^2$ so $a_{02} = 0$ and $l_3 = l$.

Equating coefficients of λ in (4) yields

$$\begin{aligned}(19)\qquad &\frac{z}{a} q a\overline{zl} + \frac{w}{b} q(b\overline{wl} + br\overline{l_2 l_4}) + rwl_1 l_2^*(\overline{l_2^* l_4}) \\ &+ a|z|^2 l + b\bar{w}(bwl + brl_2 l_3) = q\bar{l} + l \quad \text{when } |z|^2 + |w|^2 = 1.\end{aligned}$$

The first and second terms on the left clearly cancel with the first term on the right. Now setting $w = 0$, $|z| = 1$ yields $a^2 l(z,0) = l(z,0)$ so that either $a = 1$ or $l(z,w) = dw$ for some d. In case $a = 1$, $f_1(z,w) \equiv z$ by Lemma 2 and w is a factor of f_2 and f_3. Then, $(z,w) \to (f_2/w, f_3/w)$ is a proper map of B_2 onto B_2 hence of degree 1. Then we may assume

$$f(z,w) = \left(z, \frac{w(b+\alpha z + \beta w)}{1 + b(\alpha z + \beta w)}, \frac{w\sqrt{1-b^2}(\bar{\alpha} w - \bar{\beta} z)}{1 + b(\alpha z + \beta w)}\right)$$

where $|\alpha|^2 + |\beta|^2 = 1$ (up to composition by unitary maps in B_2 and B_3). One can then check to see that $Df(-\bar{\alpha} b, b\bar{\beta}\sqrt{1-b^2|\alpha|^2}/(\sqrt{1-b^2|\alpha|^2} + \sqrt{1-b^2})$ has rank one contradicting our assumption.

Thus, we assume $l(z,w) = dw$ and (19) becomes

$$(20)\qquad qr\overline{l_2 l_4} + rl_1|l_2^*|^2 \bar{l}_4 + a|z|^2 d + b^2|w|^2 d + b^2 \frac{\bar{w}}{w} rl_2 l_4 = d.$$

As previously remarked, w does not divide l_2 (this would reduce to a previous case) so w divides l_4.

Setting $w = 0$ then yields $a = 1$ or $d = 0$. Since $a = 1$ was completed above, we assume $d = 0$ so $l = 0$ and $S = 1 + l_1 l_2$. Since we now know w is a factor of f_2 and of f_3, $f_1(z,w) = f_1(z,0)$ by Lemma 2. Hence $a = 1$ (already considered) or $l_1 l_2 = az^2$. Thus, we assume

$$f(z,w) = \frac{1}{1+az^2}\left(z(a+z^2), w\left(b + cz + \frac{a}{b}z^2\right), raw^2 z + \frac{c}{br}w^2\right).$$

Again using $\langle T, Q\rangle + \langle Q, L\rangle = q\bar{l} + l$, $|z|^2 + |w|^2 = 1$, we get $a\bar{c}/b + cb = 0$. This is only possible when $c = 0$ since $0 < b < a < 1$.

On ∂B_2, the sixth degree terms of $|P|^2 - |S|^2$ are now

$$|z|^6 - |z|^2|w|^4 + \frac{a^2}{b^2}|w|^2|z|^2(|z|^2 + |w|^2) = |z|^4 - |z|^2|w|^2 + \frac{a^2}{b^2}|z|^2|w|^2.$$

Then the fourth degree terms are $(1 - a^2)|z|^4 + (a^2/b^2 - 1)|z|^2|w|^2$, and thus it must be true that $1 - a^2 = a^2/b^2 - 1$. This is $b^2 = a^2/(2 - a^2)$. The second degree terms now become

$$(1 - a^2)|z|^2 + a^2|z|^2 + b^2|w|^2 = |z|^2 + b^2|w|^2.$$

Therefore $1 = b = a$ and f is equivalent to (1)(i).

Finally, we consider case (iii). Set $q(z, w) = \alpha z^2 + \beta zw + \gamma w^2 = l_1 l_2$ so that $l_1^* l_2^* = \bar{\alpha} w^2 - \bar{\beta} zw + \bar{\gamma} z^2$. Note that $l_1^*(z, w) = 0$ and $l_1(z, w) = 0$ for the same (z, w) implies $(z, w) = 0$ unless $l_1 \equiv 0$. Further, if l_1^* and l_2 have the same zero set, then so do l_2^* and l_1, so that $l_1 l_2 = l_1^* l_2^*$ (up to multiplication by a constant) and this would again be case (i) which is complete. Also, if z divides l_1 then w divides l_1^* so $z l_1^* l_2^* = w l_1 l_2^*$ (up to multiplication by a constant) and this is case (ii). Thus, we may assume

$$q(z, w) = \alpha z^2 + \beta zw + \gamma w^2 \quad \text{and} \quad q^*(z, w) = \bar{\alpha} w^2 - \bar{\beta} zw + \bar{\gamma} z^2$$

have no common factors and $\gamma \neq 0$. Proceeding as before, using the fifth degree terms of (7),

$$Q_1 \frac{wq^*}{a} - Q_2 \frac{zq^*}{b} + Q_3 rwq = 0.$$

It clearly follows that w is a factor of Q_2 and that $Q_3 = cq^*$ for some c. Set $Q_2 = bwl_3$. Then $Q_1 = azl_3 - acrq = azl + \delta w^2$ using the minimal property of $\|Df(0)\|$. Then $\delta = -acr\gamma$ and

$$l_3 = l + cr\left(\frac{q - \gamma w^2}{z}\right) = l + cr(\alpha z + \beta w).$$

We will take $l = dz + ew$.

Proceeding as before $\langle T, Q\rangle + \langle Q, L\rangle = q\bar{l} + l$ on ∂B_2 so that

$$\begin{aligned} &|z|^2 q\bar{l} + \frac{z}{a} q\bar{\delta}\bar{w}^2 + |w|^2 q\bar{l} + |w|^2 q\bar{c}r(\overline{\alpha z + \beta w}) \\ &\quad + rz\bar{c}|q^*|^2 + a^2|z|^2 l + a\delta w^2 \bar{z} + b^2|w|^2 l \\ &\quad + b^2 cr|w|^2(\alpha z + \beta w) = q\bar{l} + l \quad \text{when } |z|^2 + |w|^2 = 1. \end{aligned}$$

The first and third terms on the left cancel with the first term on the right so (replacing $\bar{z}$ by $(1-w\bar{w})/z$) we get

$$\begin{aligned}&\frac{z}{a}q\bar{\delta}\bar{w}^2+|w|^2q\bar{c}r\left(\bar{\alpha}\frac{1-w\bar{w}}{z}+\bar{\beta}\bar{w}\right)\\&\quad+r\bar{c}z(\bar{\alpha}w^2-\bar{\beta}zw+\bar{\gamma}z^2)\left(\alpha\bar{w}^2-\beta\bar{w}\frac{1-w\bar{w}}{z}+\gamma\frac{1-2|w|^2+|w|^4}{z^2}\right)\\&\quad+(b^2-a^2)|w|^2l+a\delta w^2\frac{1-w\bar{w}}{z}\\&\quad+b^2cr|w|^2(\alpha z+\beta w)=(1-a^2)l.\end{aligned}$$

Since both sides are holomorphic in z, $z\neq 0$, they must be equal for all $z\neq 0$. The terms that have $\bar{w}$ as a factor (but not w or $\bar{w}^2$) are $-r\bar{\gamma}\beta\bar{c}z^2=0$. Since $\gamma\neq 0$ and $r\neq 0$, either $\beta=0$ or $c=0$. However, $c=0$ implies $\delta=0$ and $(b^2-a^2)|w|^2=1-a^2$ so $a=1=b$. Hence we must have $\beta=0$. The w terms are $-\bar{\beta}r\gamma=(1-a^2)e$. Since $a\neq 1$, $\beta=0$, we get $e=0$. The coefficient of $1/z$ is now

$$\bar{\alpha}\bar{c}r\gamma(|w|^2-|w|^4)w^2+r\bar{c}\bar{\alpha}\gamma(1-2|w|^2+|w|^4)w^2-a^2cr\gamma(1-|w|^2)w^2=0.$$

Thus, it follows that $r\bar{c}\bar{\alpha}\gamma=a^2cr'\gamma$ from which $\alpha=\bar{c}a^2/c=a^2$ (because we may assume $c>0$ by rotation).

The coefficient of $\bar{w}^2$ is $\alpha z^3/a+r\alpha\bar{c}\bar{\gamma}z^3=0$ so that $rc\gamma=-1/a$. That is $\gamma=-1/arc$ and we observe that $\delta=1$. Now setting $z=0$, $|w|=1$, we observe that

$$\|f(0,w)\|^2=\frac{1}{|1+\gamma w|^2}\left[|w|^2+\left|bw+\frac{\gamma w^3}{b}\right|^2+|c\bar{\alpha}w^2|^2\right]=1$$

so that

$$1+b^2+2\operatorname{Re}\gamma w^2+\frac{|\gamma|^2}{b^2}+|c|^2|\alpha|^2=1+2\operatorname{Re}\gamma w^2+|\gamma|^2.$$

Hence $b^2+|\gamma|^2/b^2+|c|^2|\alpha|^2=|\gamma|^2$. This is not possible because $|\gamma|^2/b^2>|\gamma|^2$.

Our proof is now complete.

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